Homework 1: Solutions

1.2.12 : **Problem:** Let $\mathbf{v} = (2, 3)$. Suppose $\mathbf{w} \in \mathbb{R}^2$ is perpendicular to $\mathbf{v}$, and that $\|\mathbf{w}\| = 5$. This determines $\mathbf{w}$ up to sign. Find one such $\mathbf{w}$.

Solution: If $\mathbf{w} = (a, b)$, by orthogonality to $\mathbf{v}$, we get, $\mathbf{v} \cdot \mathbf{w} = 2a + 3b = 0$. This means $a = -\frac{3b}{2}$. So $\mathbf{w} = (-\frac{3b}{2}, b)$ for some value of b.

Using the fact that $\|\mathbf{w}\| = 5$, we can figure out b:

$$5 = \|\mathbf{w}\| = \sqrt{\frac{9b^2}{4} + b^2} = \sqrt{\frac{13b^2}{4}}.$$

Squaring, we get, $b^2 = \frac{100}{13}$. So $b = \pm\frac{10}{\sqrt{13}}$. Thus the two possible answers are, $\mathbf{w} = \pm(-\frac{15}{\sqrt{13}}, \frac{10}{\sqrt{13}})$.

1.2.18 : **Problem:** Find all values of x such that $(x, 1, x)$ and $(x, -6, 1)$ are othogonal.

Solution: To be orthogonal, their dot product must be 0.

That is, $(x, 1, x) \cdot (x, -6, 1) = x^2 - 6 + x = 0$.

This quadratic equation factors as: $(x + 3)(x - 2) = 0$, so the only two values are $x = -3, 2$, for which the two vectors will be orthogonal.

1.3.16 : **Problem:** Find an equation for the plane that passes through:

(a) $(0, 0, 0), (2, 0, -1)$, and $(0, 4, -3)$.

(b) $(1, 2, 0), (0, 1, -2)$, and $(4, 0, 1)$.

(c) $(2, -1, 3), (0, 0, 5)$, and $(5, 7, -1)$.

Solution

(a): Let $P = (0, 0, 0), Q = (2, 0, -1)$, and $R = (0, 4, -3)$

Then the vectors $\overrightarrow{PQ} = (2, 0, -1)$, and $\overrightarrow{PR} = (0, 4, -3)$ are on the plane. A normal vector to this plane is $\overrightarrow{n} = (2, 0, -1) \times (0, 4, -3)$, which we calculate as follows:

$$\vec{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 0 & 4 & -3 \end{vmatrix} = (0-(-4))\mathbf{i} - (-6-0)\mathbf{j} + (8-0)\mathbf{k} = 4\mathbf{i} + 6\mathbf{j} + 8\mathbf{k}.$$

So the equation of the plane must be of the form, $4x+6y+8z+D=0$. And since $P=(0,0,0)$ is on the plane, it must satisfy this equation, so plugging in $(x,y,z)=(0,0,0)$, we get $D=0$.

Thus an equation of the plane passing through $(0,0,0),(2,0,-1)$, and $(0,4,-3)$ is $4x+6y+8z=0$.

(b): Let $P=(1,2,0), Q=(0,1,-2)$, and $R=(4,0,1)$

Then the vectors $\overrightarrow{PQ}=(-1,-1,-2)$, and $\overrightarrow{PR}=(3,-2,1)$ are on the plane. A normal vector,

$$\vec{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -1 & -2 \\ 3 & -2 & 1 \end{vmatrix} = (-1-(4))\mathbf{i} - (-1-(-6))\mathbf{j} + (2-(-3))\mathbf{k} = -5\mathbf{i} - 5\mathbf{j} + 5\mathbf{k}.$$

So the equation of the plane must be of the form, $-5x-5y+5z+D=0$. And since $P=(1,2,0)$ is on the plane, it must satisfy this equation, so plugging in $(x,y,z)=(1,2,0)$, we get $-5(1)-5(2)+D=0$. So $D=15$

Thus an equation of the plane passing through$(1,2,0),(0,1,-2)$, and $(4,0,1)$ is $-5x-5y+5z+15=0$, or $x+y-z=3$.

(c): Let $P=(2,-1,3), Q=(0,0,5)$, and $R=(5,7,-1)$

Then the vectors $\overrightarrow{PQ}=(-2,1,2)$, and $\overrightarrow{PR}=(3,8,-4)$ are on the plane.

$$\vec{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 2 \\ 3 & 8 & -4 \end{vmatrix} = (-4-(16))\mathbf{i} - (8-6)\mathbf{j} + (-16-3)\mathbf{k} = -20\mathbf{i} - 2\mathbf{j} - 19\mathbf{k}.$$

So the equation of the plane must be of the form, $-20x - 2y - 19z + D = 0$. And since $Q = (0, 0, 5)$ is on the plane, $-19(5) + D = 0$. So $D = 95$

Thus an equation of the plane passing through$(2, -1, 3), (0, 0, 5)$, and $(5, 7, -1)$ is $-20x - 2y - 19z + 95 = 0$, or $20x + 2y + 19z = 95$.

1.3.22 : **Problem:** Find the intersection of the two planes with equations
$3(x-1) + 2y + (z+1) = 0$ and
$(x-1) + 4y - (z+1) = 0$

Solution: The coefficients of x, y, z do not form parallel vectors, so the two planes must not be parallel. Thus they must intersect in a line. We will find a parametric form of their line of intersection. We have,

$4(x-1) + 6y = 0$ – by adding the two equations, (to eliminate z), and

$-10y + 4(z+1) = 0$ – by subtracting 3 times the second from the first, (to eliminate x).

So

$$z = -1 + \frac{10y}{4}$$

$$x = 1 - \frac{6y}{4}$$

Now, by setting $y = t$ (a parameter), we obtain that the line of intersection must be:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 - \frac{6t}{4} \\ t \\ -1 + \frac{10y}{4} \end{pmatrix}$$

This could also be written as,

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + t \begin{pmatrix} -3/2 \\ 1 \\ 5/2 \end{pmatrix}, \text{ or } \begin{pmatrix} 1-3s \\ 2s \\ 5s-1 \end{pmatrix}, \text{for } s = t/2.$$

1.4.4 : **Problem:** Describe the geometric meaning of the following mappings in cylindrical coordinates:

(a) $(r, \theta, z) \mapsto (r, \theta, -z)$.

(b) $(r, \theta, z) \mapsto (r, \theta + \pi, -z)$.

(c) $(r, \theta, z) \mapsto (-r, \theta - \pi/4, z)$.

Solution:

(a): Since r, θ remain unchanged, only the height along the z-axis of a point is changed. Each point is reflected across the xy-plane (or $r\theta$-plane), as if there were a mirror on this plane. Points on the xy-plane remain fixed.

(b): Adding π to θ rotates each point about the z-axis by 180 degrees. This changes the (x, y) values to $(-x, -y)$.

Simultaneously, since z is changed to $-z$, the resulting mapping takes (x, y, z) to $-(x, y, z)$. Thus this mapping amounts to "reflecting across the origin", or simply negating every vector.

(c): If r is not allowed to be negative, this mapping does not make sense, and points were not removed for stating this.

However, if r is allowed to be negative to describe mappings, then $r \mapsto -r$ has the same affect as $\theta \mapsto \theta + \pi$, or of rotating about the z-axis by 180 degrees. $\theta \mapsto \theta - \pi/4$ has the affect of rotating about the z-axis clockwise (looking from above) by 45 degrees. All in all this mapping rotates about the z-axis counterclockwise by $180 - 45 = 135$ degrees.

1.4.10 : **Problem:** Describe the following solids using inequalitites. State the coordinate system used.

(a) A cylindrical shell 8 units long, with inside diameter 2 units and outside diameter 3 units

(b) A spherical shell with inside radius 4 units and outside radius 6 units

(c) A hemisphere of diameter 5 units

(d) A cube of side length 2

Solution:

(a): This is best described in cylindrical coordinates. Note for diameter = 2 use radius = 1.

$$1 \le r \le 3/2 \text{ and } |z| \le 4 \text{ (one possibility)}$$

(b): This is best described in spherical coordinates.

$$4 \le \rho \le 6$$

(c): This is also best described in spherical coordinates.

$$\rho \le 5/2 \text{ and } 0 \le \theta \le \pi$$

or

$$\rho \le 5/2 \text{ and } 0 \le \phi \le \pi/2$$

Note, $0 \le \phi \le \pi$ would give the whole sphere. If the sphere is assumed hollow, use $\rho = 5/2$ instead of $\rho \le 5/2$.

(d): This is best described in Cartesian (or rectangular) coordinates as:

$$|x| \le 1$$
$$|y| \le 1$$
$$|z| \le 1$$